Chapter 1
An Overview of B-branes in Gauged Linear Sigma Models

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Abstract We review the BPS D-branes in gauged linear sigma models corresponding to toric Calabi-Yau (CY) varieties preserving $\mathscr{N}=2_B$ supersymmetry, and their relation to stable low energy branes. The chiral sectors of these low energy branes are described mathematically by various derived categories in various parts of the CY Kähler moduli space $\mathscr{M}_K$. For a fixed $\mathscr{M}_K$, all these descriptions should in fact be equivalent in a categorical sense and we review some aspects of this equivalence from a physical perspective. This is a short summary of the results of the comprehensive work by Herbst, Hori and Page [HHP08] with some elementary commentary.

1.1 Introduction

Two dimensional superconformal field theories (SCFTs) with $\mathscr{N}=(2,2)$ supersymmetry have been a topic of much investigation for a long time, partly due to their essential role in string theory. Some natural objects to study in any field theory are boundaries in the space-time manifold preserving various amount of symmetry. In case of supersymmetric theories, in the presence of boundaries, at most half of the supersymmetry can be preserved. The boundaries preserving this maximal amount of supersymmetry are known as BPS D-branes. The BPS D-branes in $\mathscr{N}=(2,2)$ SCFTs are particularly interesting objects as they provide the best (most manageable) probes of "stringy" geometry that we have so far. In particular, they have played a central role in our understanding of string dualities.[1] There are

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[1] Of particular relevance to the topic of this review is the mirror symmetry, where D-branes underlie both the homological approach of Kontsevich [Kon94] and the T-duality perspective of Strominger, Yau, and Zaslow [SYZ96].

two inequivalent subalgebras of $\mathcal{N} = (2,2)$ superconformal algebras (SCAs) containing half of the supersymmetry, they are known as $\mathcal{N} = 2_A$ and $\mathcal{N} = 2_B$ SCAs, and the BPS D-branes that preserve these subalgebras are called the *A-branes* and the *B-branes* respectively. The principal objects of study in this review are the B-branes.

The $\mathcal{N} = (2,2)$ SCFTs have a moduli space with a product structure: $\mathcal{M}_{\text{SCFT}}^{\mathcal{N}=(2,2)} = \mathcal{M}_K \times \mathcal{M}_C$, where $\mathcal{M}_K$ and $\mathcal{M}_C$ correspond to the Kähler and complex structure moduli of some Calabi-Yau (CY) manifold . The infinitesimal deformations (preserving superconformal symmetry) of these SCFTs are generated by exactly marginal operators which correspond to the tangents to these moduli spaces. It is usually difficult, beyond perturbation theory, to study the algebraic structures of operators in full generality of a quantum field theory (QFT). But, the exactly marginal operators corresponding to $\mathcal{M}_K$ and $\mathcal{M}_C$ belong to two special subsectors of operators, known respectively as the twisted chiral ring ($\mathscr{R}_{\text{tc}}$) and the chiral ring ($\mathscr{R}_{\text{c}}$), that can be studied in isolation with exact precision . These rings, $\mathscr{R}_{\text{c}}$ and $\mathscr{R}_{\text{tc}}$, are independent of Kähler deformations and complex structure deformations respectively. Another defining feature of $\mathscr{R}_{\text{c}}$ and $\mathscr{R}_{\text{tc}}$ is that they correspond to the $\boldsymbol{Q}$-cohomology of operators for a supercharge $\boldsymbol{Q}$ in $\mathcal{N} = 2_B$ and $\mathcal{N} = 2_A$ algebras respectively. This leads us to the fact that the *boundary chiral ring* and the *boundary twisted chiral ring*[2] contain some supersymmetry invariant data about the B-branes and the A-branes respectively. These supersymmetric data are of particular interest due to the fact that they are protected under renormalization (RG) , which facilitates many exact computations. Furthermore, the chiral (twisted-chiral) sector of the B-branes (A-branes) should be independent of the Kähler (complex structure) moduli and should vary with the complex structure (Kähler) moduli.

The independence of the boundary chiral ring from the Kähler moduli in presence of a B-brane does not mean that the B-branes are completely unaffected by a change in the Kähler moduli, what it means is that the chiral sectors of the B-branes over different points of the Kähler moduli space are *isomorphic* to each other,[3] often quite non-trivially. Finding a unifying description of the chiral sectors of the B-branes over the bulk of the Kähler moduli space was the principal motivation and result of the comprehensive work [HHP08] and is the topic of this review.

A priori, it is a difficult task to relate the B-branes at different points of the Kähler moduli space $\mathcal{M}_K$, because at different points, the low energy theory can look drastically different with different looking descriptions for the B-branes which are not trivially identified. For example, in some region of $\mathcal{M}_K$, the theory may be described by a non-linear sigma model on a non-compact Calabi-Yau and in another region by an orbifold theory with a discrete gauge group . The B-branes in these two theories are described as objects in some derived categories associated to the Calabi-Yau and the orbifold. The fact that these two categories are equivalent is a non-trivial statement which is familiar in the math literature as categorical McKay correspondence [Kaw03, BKR99]. Another example of such equivalence is when the low energy

[2] Elements of the chiral and the twisted chiral rings that can be placed at a boundary.

[3] In a categorical sense, which we will make a bit clearer over the course of the review.

theory at two different regions are described by a non-linear sigma model on a compact Calabi-Yau and a Landau-Ginzburg (LG) type theory. The relevant categorical equivalence in this case, between a derived category of CY hypersurfaces defined by some polynomial and the category of matrix factroization of the same polynomial, was established by Orlov [Orl05]. Explaining these equivalences from a physical definition of the B-branes and their chiral sectors was part of the motivation for the paper [HHP08].

The main tool in overcoming the obstacle of having different looking theories at different places in $\mathscr{M}_K$, is to use an ultraviolet (UV) description of the theories, called the gauged linear sigma model (GLSM) [Wit93], which includes as parameters the coordinates of $\mathscr{M}_K$. Depending on the values of these parameters it flows to the various infrared (IR) descriptions under renormalization. Therefore, having a description of the B-branes in the GLSM allows to see the fate of these B-branes under RG flow and makes the connection between the IR B-branes in different regions of $\mathscr{M}_K$ physically transparent.

In this spirit we begin, in §1.2 by reviewing some basic facts about GLSMs and their low energy description in the absence of boundary. All of this and much more was discussed at length by Witten in [Wit93]. In §1.3 we review the characterization of $\mathscr{N}=2_B$ preserving boundaries in GLSM and the boundary chiral rings. We will see that these branes can be thought of as objects in a homotopy category of graded modules for certain rings and the chiral ring elements correspond to the morphisms in the category. In the last section, §1.4, we review the identification of the GLSM B-branes and the IR B-branes, and the mechanism of relating the chiral sectors of these IR B-branes in different regions of $\mathscr{M}_K$. It turns out that the relation between the GLSM B-branes and the IR B-branes is many to one and the IR branes are better described as objects in some derived category. These last two sections comprise a much abridged summary of a portion of the extensive work [HHP08]. In the appendix we collect some background information about the $\mathscr{N}=(2,2)$ supersymmetry algebra and some general remarks about Chan-Paton spaces relevant for describing the branes.

We should mention as a disclaimer that the main focus of this review is to point out the emergence of some mathematical notions purely from physical considerations. We will not go into much details of the relevant mathematics and we will often be rather schematic. A good (physics friendly) reference for much of the mathematics involved is [ABC$^+$09].

1.2 GLSM without boundary

On a $(1+1)$D space-time Σ (also called the *world-sheet*), $\mathscr{N}=(2,2)$ GLSMs are supersymmetric QFTs of maps:

$$x:\Sigma\to\mathbb{C}^N\,, \tag{1.1}$$

with some gauge group T. Using coordinates for $\mathbb{C}^N$, the map x can be thought of as a collection of N maps $x = (x^1, \cdots, x^N)$ where for any $p \in \Sigma$, $x^i(p)$ is the i'th coordinate of $x(p)$. We will only consider compact abelian gauge groups $T \cong U(1)_1 \times \cdots \times U(1)_k$. Let us denote the charge of x^i under $U(1)_a$ by Q_i^a, i.e., an arbitrary element $g := (e^{i\phi_a}, \cdots, e^{i\phi_k}) \in T$ acts on the fields (1.1) as:

$$g : x^i \mapsto e^{i\sum_{a=1}^k Q_i^a \phi_a} x^i \,. \tag{1.2}$$

Note that for $k \geq N$, a $(k-N)$-dimensional subalgebra of the gauge algebra $\bigoplus_{a=1}^k u(1)_a$ acts trivially on the fields, therefore without loss of generality we assume $k \leq N$. Supersymmetry introduces superpartners for the fields x^i and we end up with *chiral multiplets* $\Phi^i = (x^i, \psi^i, F^i)$ where ψ^i's are Dirac fermions and F^i's are complex scalars. Similarly the gauge fields for the gauge group sit in vector multiplets $V_a = (\sigma_a, v_{a,\mu}, \lambda_a, D_a)$ for $a \in \{1, \cdots, k\}$, where σ_a are complex scalars, v_a is the 1-form gauge field for $U(1)_a$, λ_a is a Dirac fermion, and D_a's are real scalars. If there exists a gauge invariant holomorphic polynomial $W : \mathbb{C}^n \to \mathbb{C}$ then this can be included as additional data defining the theory. With all these data given, and upon defining the *field strength superfield*s $\Sigma_a := \overline{D}_+ D_- V_a$ (which are twisted chiral), the Lagrangian of the theory, *in the absence of space-time boundary*, can be written most conveniently as a sum of several superspace integrals (Berezin integrals):

$$\begin{aligned}\mathscr{L}_{\text{bulk}} = \int \mathrm{d}^2\theta \mathrm{d}^2\overline{\theta} \left(-\frac{1}{2} \sum_{a,b=1}^k (e^{-2})_{ab} \overline{\Sigma}_a \Sigma_b + \sum_{i=1}^N \overline{\Phi}^i e^{\sum_{a=1}^k Q_i^a V_a} \Phi^i \right) \\ + \Re \int \mathrm{d}\theta^+ \mathrm{d}\overline{\theta}^- \left(-\sum_{a=1}^k t^a \Sigma_a \right) + \Re \int \mathrm{d}\theta^+ \mathrm{d}\theta^- \, W(\Phi) \,.\end{aligned} \tag{1.3}$$

In order to write the above Lagrangian we have introduced the gauge couplings e_{ab}^2 (which are real numbers) and (e^{-2}) refers to the inverse of the $k \times k$-matrix e_{ab}^2. We have also introduced the complexified *Fayet-Iliopoulos* (FI) parameters:

$$t^a := r^a - \frac{i\theta^a}{2\pi} \,, \tag{1.4}$$

where r^a's are called the *real* FI parameters and θ^a's the topological theta angles.

1.2.1 Symmetries

The Lagrangian (1.3) has manifest $(2,2)$ supersymmetry since it is composed of $(2,2)$ superspace integrals.[4] Other global symmetries include space-time isometry

[4] Which implies that the supersymmetry variation of this Lagrangian is a total derivative, and in the absence of boundaries the action is invariant.

(the Lorentz symmetry) $U(1)_L$ and an *axial* $U(1)_A$ R-symmetry.[5] If the superpotential is quasi-homogeneous, i.e., if it satisfies:

$$W(\xi^{p_i}x^i) = \xi^2 W(x^i)\,, \tag{1.5}$$

for some numbers $p_i \in \mathbb{R}$, then there is a *vector* $U(1)_V$ R-symmetry as well. The action of these symmetry groups and the gauge group can be summerized by mentioning the respective charges of various objects:

$$\begin{array}{c||c|c||c|c} & \Phi^i & \Sigma & \theta^\pm & \overline{\theta}^\pm \\ \hline U(1)_L & 0 & 0 & \pm & \pm \\ U(1)_a\,,\ a\in\{1,\cdots,k\} & Q^a_i & 0 & 0 & 0 \\ U(1)_V & p_i & 0 & + & - \\ U(1)_A & 0 & 2 & \pm & \mp \end{array} \tag{1.6}$$

Though the axial R-symmetry is always present classicaly, it is generically broken quantum mechanically due to anomaly of the fermion (of the chiral multiplets) measure :

$$\delta_{u(1)_A}\left(\prod_{i=1}^{N}\mathscr{D}\psi^i\mathscr{D}\overline{\psi}^i\right) \propto \sum_{a=1}^{k}\sum_{i=1}^{N} Q^a_i c_1(E_a)\prod_{i=1}^{N}\mathscr{D}\psi^i\mathscr{D}\overline{\psi}^i\,, \tag{1.7}$$

where E_a denotes the $U(1)_a$-bundle over Σ and $c_1(E_a)\in\mathbb{Z}$ refers to its first Chern number. Note that, the twisted chiral superspace integral from the Lagrangian 1.3, in terms of the component fields, become:

$$\Re\int \mathrm{d}\theta^+\mathrm{d}\overline{\theta}^-\left(-\sum_{a=1}^{k}t^a\Sigma_a\right) = -\sum_{a=1}^{k}\left(r^aD_a + \frac{\theta^a}{2\pi}v_{a,01}\right)\,, \tag{1.8}$$

where $v_{a,01} := \partial_0 v_{a,1} - \partial_1 v_{a,0}$ is the gauge field strength. Since

$$\frac{1}{2\pi}\int_\Sigma \mathrm{d}^2y\, v_{a,01} = c_1(E_a)\in\mathbb{Z}\,, \tag{1.9}$$

and in a path integral θ-angles will only appear in exponentials

$$\exp\left(\frac{i\theta_a}{2\pi}\int \mathrm{d}^2y\, v_{a,01}\right)\,, \tag{1.10}$$

the θ-angles are truly angular variables, $\theta_a \sim \theta_a + 2\pi$. Furthermore, we see that a change of the θ-angles is equivalent to a field redefinition via an axial R-rotation when (1.7) is nonzero. Therefore, in such cases the θ-angles are not actual parameters of the theory. They become true parameters when the following condition is met:

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[5] An *R-symmetry* is a symmetry that acts nontrivially on the fermionic coordinates of the superspace and is not part of the space-time isometry.

$$\sum_{i=1}^{N} Q_i^a = 0 \quad \forall a \in \{1, \cdots, k\}. \tag{1.11}$$

This condition is called the *Calabi-Yau condition* because when this condition holds, for some range of the FI parameters, the linear sigma model flows in the infrared to a non-linear sigma model with a Calabi-Yau target space . Unlike the axial R-symmetry, the vector R-symmetry is anomaly free whenever it exists as a classical symmetry (i.e., when the superpotential is quasi-homogeneous) .

The $\mathscr{N} = (2,2)$ superconformal algebra necessarily includes both the axial and the vector R-symmetry [SS87]. Therefore, unless both of these R-symmetries exist as quantum symmetries in the UV GLSM, it will not flow to a non-trivial IR conformal fixed point. Since our primary goal is to describe B-branes in the IR SCFTs, we will only consider GLSMs with quasi-homogeneous superpotentials (1.5) and gauge charges satisfying the CY condition (1.11) to ensure that both of the R-symmetries are preserved quantum mechanically.

1.2.2 Low energy phases

Performing the superspace integrals we get the Lagrangian in terms of the component fields . Then we find that the component fields D_a and F^i do not have any kinetic terms and therefore in a path integral we can perform the integrations over these fields, the outcome of which is simply to replace them by solving their classical equations of motion, which are algebraic.[6] These fields are called auxiliary fields and after integrating them out the classical potential for the rest of the bosonic scalars become:

$$U(\sigma, x) := \sum_{i=1}^{N} \left| \sum_{a=1}^{k} Q_i^a \sigma_a x^i \right|^2 + \frac{e^2}{2} \sum_{a=1}^{k} \left(\sum_{i=1}^{N} Q_i^a |x^i|^2 - r^a \right)^2 + \sum_{i=1}^{N} \left| \frac{\partial W(x)}{\partial x^i} \right|^2, \tag{1.12}$$

where $x = (x^1, \cdots, x^N) \in \mathbb{C}^N$ and we have assumed a simple form of the gauge couplings $e_{ab}^2 = e^2 \delta_{ab}$ which makes no qualitative difference in our discussion.

1.2.2.1 Classical analysis

A classical analysis of the space of vacua sheds some light on the qualitative nature of the low energy description of the theory. The classical space of vacua is the space of solutions of the equation:

$$U(\sigma, x) = 0. \tag{1.13}$$

[6] It may seem a little redundant to introduce them in the first place. They serve the purpose of closing the supersymmetry algebra (on the fields) off shell (without using equations of motion) which is of paramount importance in some cases, such as in discussing supersymmetry in curved backgrounds, but they play no significant role for us at the moment.

The qualitative nature of this space of solutions depends on the real FI parameters. Since the three terms of (1.12) are positive semi-definite, they must all separately vanish for U to vanish. Let us pick some *generic* values of the real FI parameters $r := (r^1, \cdots, r^k)$. Now we set the second term in (1.12) to zero (the resulting equations are called the *D-term* equations):

$$\sum_{i=1}^{N} Q_i^a |x^i|^2 = r^a\,, \quad \forall a \in \{1, \cdots, k\} \tag{1.14}$$

The space of solutions to this system of equations is (real) $(N-k)$ dimensional and therefore at most $(N-k)$ of the x^i's can be simultaneously zero (due to the generic nature of the r^a's). By setting the first term of (1.12) to zero we find the following equations:

$$\sum_{a=1}^{k} Q_i^a \sigma_a = 0 \quad \forall i \in \{1, \cdots, N\} \quad \text{such that} \quad x^i \neq 0\,. \tag{1.15}$$

Thus we have a homegenous system of at least k linear equations in k variables, the only solution being:

$$\sigma_a = 0 \quad \forall a \in \{1, \cdots, k\}\,. \tag{1.16}$$

This shows that for generic r, the gauge group is either completely broken or reduces to some discrete subgroup at low energy (i.e. no continuous degree of freedom left in the gauge multiplets). Note that not all of the solutions of (1.14) are physically distinguishable since some of them can be related by the action of the gauge group. Therefore, we should consider the quotient space:

$$X_r := \left\{ (x^1, \cdots, x^N) \in \mathbb{C}^N \,\middle|\, \sum_{i=1}^{N} Q_i^a |x^i|^2 = r^a \right\} \Big/ T = \mathbb{C}^N // T\,. \tag{1.17}$$

This is a symplectic quotient of $\mathbb{C}^N$ by T, since the D-term equations (1.14) can be interpreted as setting the moment maps of the T-action on $\mathbb{C}^N$ to zero. Equivalently, X_r can be written as a *toric variety*, i.e., an ordinary quotient of $\mathbb{C}^N$ (minus some "bad" points) by the *complexified* gauge group (a complex algebraic torus):

$$X_r = \left(\mathbb{C}^N \backslash \Delta_r\right) \big/ T_{\mathbb{C}}\,, \tag{1.18}$$

where the *deleted set* Δ_r is the set of points whose $T_{\mathbb{C}}$-orbits do not pass through the space of solutions to the D-term equations (1.14). In this form it is clear how the dependence on the real FI parameters enter into the determination of the classical space of vacua, the real FI parameters determine the deleted set. Finally by setting the last term of the potential (1.12) to zero we find that the classical space of vacua is given by the intersection:[7]

[7] The equations $dW = 0$ are called the *F-term* equations. Also note that, as defined, the classical space of vacua $\mathrm{Vac}_r^{\mathrm{cl}}$ has a metric induced from the Fubiny-Study type metric on the toric variety X_r. If we could add all quantum corrections corresponding to integrating out all the massive modes,

$$\mathrm{Vac}_r^{\mathrm{cl}} := X_r \cap \left\{ x = (x^1, \cdots, x^N) \in \mathbb{C}^N \,\middle|\, \mathrm{d}W(x) = 0 \right\} \tag{1.19}$$

The vacuum configurations where fields from chiral multiplets[8] acquire non-zero values, such as in the above description, are referred to as the *Higgs branch* of vacua. These vacua are characterized by the possibility of breaking the gauge group completely, or to some discrete subgroup.[9]

The geometry[10] of $\mathrm{Vac}_r^{\mathrm{cl}}$ varies smoothly with varying r, except when r falls into some (real codimension one) hypersurfaces where solutions to the D-term equations (1.14) include more than $(N-k)$ of the x^i's vanishing. In such non-generic cases, some of the σ_a's become unconstrained. Such hypersurfaces are invariant under scaling,[11] so the space of real FI parameters, let it be denoted by $\mathbb{R}^k_{\mathrm{FI}}$, becomes divided into cones, the cones are called *phases* of the low energy theory. The boundaries of these cones, i.e. the hypersurfaces, are called *phase boundaries* or *singular loci*.[12] At least classically, we can expect the nature of the low energy theory to be qualitatively the same in a given phase, but the nature changes significantly across phases.

If $W = 0$, then all excitations transverse to the classical space of vacua $\mathrm{Vac}_r^{\mathrm{cl}} = X_r$ acquire masses from the second term of the potential (1.12), so the low energy theory consists of excitations tangential to X_r. In the absence of any residual discrete gauge symmetry, such theories are known as non-linear sigma models with target X_r. If some discrete gauge symmetry Γ survives and the space of vacua can be realized as a global orbifold $X_r = X'_r/\Gamma$, the low energy theories are known as orbifold theories [DHVW85]: Γ-gauged sigma models on X'_r.

In the presence of a non-trivial superpotential, there are phases where some excitations tangential to X_r but transverse to $\mathrm{Vac}_r^{\mathrm{cl}}$ picks up masses from the superpotential. The extreme example of such cases is when all such excitations become massive, i.e., only excitations tangential to $\mathrm{Vac}_r^{\mathrm{cl}}$ remain massless, these are called *purely geometric* phases or non-linear sigma models on $\mathrm{Vac}_r^{\mathrm{cl}}$. The opposite extreme is the *Landau-Ginzburg* phase where $\mathrm{Vac}_r^{\mathrm{cl}}$ consists of just a single point and all excitations of the fields x^i become massless, possibly with some discrete gauge symmetry .

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we would find that this metric (which appears in the kinetic term for the x^i's) gets modified to a Ricci-Flat (Calabi-Yau) metric [HKK$^+$03].

[8] More generally, fields that transform under faithful representations of the gauge group.

[9] Unlike fields from vector multiplets which transform under the adjoint representation and therefore can break the gauge group only upto the maximal torus.

[10] More specifically, the Kähler geometry.

[11] Since, if x is a solution of (1.14) for some r then $\sqrt{\xi}x$ is a solution of (1.14) for ξr.

[12] The singular nature arises from integrating out the σ_a's (which become massless on these loci) and trying to keep only the chiral multiplet fields as dynamical, which only works well away from the singularity.

Thus, classically, the real FI parameters $\mathbb{R}^k_{\mathrm{FI}}$ and any parameters that my appear in the superpotential, parametrize the low energy theories. The FI parameters[13] become the Kähler moduli, denoted as $\mathscr{M}_K$, of the target space for the low energy theory and the superpotential paratemeters become the complex structure moduli, denoted as $\mathscr{M}_C$. As we have seen, the classical Kähler moduli space is divided into disconnected phases by the singular loci. This picutre changes significantly with quantum corrections.

1.2.2.2 Quantum corrections

Quantum mechanically the space of low energy theories depend in addtion to the real FI parameters, on the θ-angles. In our classical analysis the phase boundaries (i.e. the singular loci) were places in the Kähler moduli space $\mathscr{M}_K$ where a Coulomb branch could open up, i.e., a complex scalar σ_a from the vector multiplets became unconstrained (massless). When this happens we can go to arbitrary large[14] values for this σ_a and this gives the chiral fields x^i large masses by the first term in the potential (1.12). So we can look at the one loop effect of integrating out these massive fields.[15] To see how the picture of the moduli space changes in the quantum theory we can assume large values for σ_a and look at its one loop effective potential $U_{\mathrm{eff}}(\sigma)$ after integrating out the x^i's. The zero locus of this potential in terms of the parameters is the quantum corrected singular loci in $\mathscr{M}_K$. We will illustrate this procedure with a concrete example and we will see that the effective potential U_{eff} depends on the complexified FI parameter $t^a = r^a - \frac{i\theta^a}{2\pi}$ and the singular regions comprise a *complex* hypersurface in the space of these complex FI parameters. Therefore, unlike the classical result where the singular regions divided the $\mathbb{R}^k_{\mathrm{FI}}$ into disconnected phases, in the quantum corrected picture the phases are all continuously connected.

Example 1. The example we consider is a $U(1)$ gauge theory with the following chiral fields with gauge charges satisfying the Calabi-Yau condition (1.11):

$$\begin{array}{c|cccc} \text{Fields} & x^1 & x^2 & x^3 & x^4 \\ \hline \text{Gauge charges, } Q_i & 1 & 1 & 1 & -3 \end{array} . \tag{1.20}$$

We consider the theory without a superpotential. There is one real FI parameter $r \in \mathbb{R}_{\mathrm{FI}}$. In constructing the space of vacua X_r for $r > 0$ and $r < 0$, the respective deleted sets are:

$$\begin{aligned} \Delta_+ &= \{x^1 = x^2 = x^3 = 0\} \qquad && \text{for } r > 0\,, \\ \Delta_- &= \{x^4 = 0\} && \text{for } r < 0\,. \end{aligned} \tag{1.21}$$

[13] Classically we can only see the effects of the real FI parameters but after including quantum corrections, the true parameters of the low energy theories will be the complexified FI parameters (1.4).

[14] Compared to the scale of theory set by the gauge coupling e, which has mass dimension 1.

[15] The one loop contribution to the effective potential is *exact* due to a non-renormalization theorem for the twisted superpotential .

At low energy the $r > 0$ phase is described by a non-linear sigma model on $X_+ := \mathscr{O}_{\mathbb{CP}^2}(-3)$ and the $r < 0$ phase is described by the orbifold theory on $X_- := \mathbb{C}^3/\mathbb{Z}_3$.[16] Classically these two phases are separated by the singularity at the origin.

For large σ, i.e. $|\sigma| \gg e$, the effect of integrating out the massive chiral fields is to introduce the following effective action for σ [Wit93, HKK+03]:

$$U_{\text{eff}}(\sigma) = \frac{e_{\text{eff}}^2}{2} \left| r_{\text{eff}} - i\theta_{\text{min}} \right|^2 . \tag{1.22}$$

where, e_{eff} is the renormalized coupling, $r_{\text{eff}} = r + \sum_i Q_i \log |Q_i|$ and $|\theta_{\text{min}}|^2 = \min_{n\in\mathbb{Z}} (\theta + s\pi + 2\pi n)^2$ with s being the sum of the positive charges, $s := \sum_{Q_i \geq 0} Q_i$. Thus, the solution to the equation $U_{\text{eff}}(\sigma) = 0$ is given by:

$$r = r_0 := -\sum_i Q_i \log |Q_i| , \qquad \theta = s\pi \,(\text{mod}\, 2\pi) . \tag{1.23}$$

So we see that the singular locus in the one complex dimensional Käler moduli space is a one complex codimensional hypersurface.

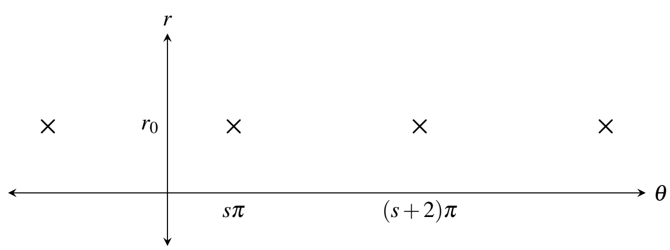

Fig. 1.1 A typical Kähler moduli space $\mathscr{M}_K$ with singularities, for some toric variety ($\dim_{\mathbb{C}} \mathscr{M}_K = 1$). The θ direction is 2π periodic.

1.3 Boundaries

Branes are subspaces of the target space where the world-sheet can end. Having boundaries of the world-sheet necessarily breaks some symmetries of the world-sheet theory. Partial symmetry can be preserved by introducing *boundary interactions*. We are interested in describing BPS D-branes preserving half ($\mathscr{N} = 2_B$ to be particular) of the world-sheet supersymmetry.

[16] For $r > 0$, the base $\mathbb{CP}^2$ is the quotient $\{\mathbb{C}^3 \backslash \Delta_+\}/\mathbb{C}^\times$ by the complexified gauge group, where the $\mathbb{C}^3$ is spanned by (x^1, x^2, x^3), and x^4 becomes the coordinate on the fiber. For $r < 0$, $\{\mathbb{C} \backslash \Delta_-\}/\mathbb{C}^\times$ is a point which is invariant under a discrete gauge group $\mathbb{Z}_3 \subset U(1)$ and (x^1, x^2, x^3) spans the $\mathbb{C}^3$ carrying a nontrivial representation of $\mathbb{Z}_3$.

In the presence of boundaries the action $S_{\text{bulk}} = \int_\Sigma \mathrm{d}^2 y \, \mathscr{L}_{\text{bulk}}$, where $\mathscr{L}_{\text{bulk}}$ is given by (1.3), is not supersymmetric since the supersymmetry variation of $\mathscr{L}_{\text{bulk}}$ is a total derivative and therefore the variation is a boundary integral. We add to it a boundary action:

$$S_{\text{bulk}} \to S = S_{\text{bulk}} + S_{\text{bdry}}\,, \qquad S_{\text{bdry}} = \int_{\partial\Sigma} \mathrm{d}y \, \mathscr{L}_{\text{bdry}}\,, \tag{1.24}$$

such that the variation of the boundary interaction cancels the variation of the bulk action for some supercharge $\mathscr{Q}$:

$$\delta_{\mathscr{Q}} S = 0\,. \tag{1.25}$$

If we can do this then the supercharge $\mathscr{Q}$ is preserved in presence of the boundary. It is only possible to preserve at most two of the four supercharges of the $\mathscr{N} = (2,2)$ algebra. In what follows we will be concerned with $\mathscr{N} = 2_B$ preserving boundaries, i.e., the B-branes. The explicit form of S_{bdry} that makes S 2_B-invariant can be found in [Hor00, HKLM02].

It turns out that the choice of boundary interaction, $\mathscr{L}_{\text{bdry}}$ in (1.24), is not unique. For two choices of boundary interactions, $\mathscr{L}^1_{\text{bdry}}$ and $\mathscr{L}^2_{\text{bdry}}$, the difference in the corresponding boundary actions, namely $\int_{\partial\Sigma} \mathrm{d}y \left(\mathscr{L}^1_{\text{bdry}} - \mathscr{L}^2_{\text{bdry}} \right)$, is necessarily 2_B-invariant on its own. Thus, different 2_B-invariant boundary actions give rise to different B-branes. In the following we discuss how to characterize these $\mathscr{N} = 2_B$ supersymmetric boundaries.

1.3.1 Chan-Paton factors

Chan-Paton factors are essentially degrees of freedom associated to boundaries of our worldsheet. Various group actions can be defined on the boundary which leads to the introduction of these degrees of freedom and eventually associates rich structures to the boundaries which describe the branes. An introduction to these Chan-Paton spaces have been included in §1.B. Here we discuss the Chan-Paton spaces describing the GLSM B-branes.

1.3.1.1 Gauge group, $U(1)^k$: Wilson line branes

In a gauge theory, a natural operator supported on a line defect or a boundary C is the Wilson line:[17]

$$\operatorname{Tr} W_\rho(C)\,, \qquad \text{with} \quad W_\rho(C) := \mathscr{P} \exp\left(i \int_C \rho_*(A) \right), \tag{1.26}$$

[17] The symbol $\mathscr{P}$ means *path ordered* which is a prescription to make sense of the exponential of the integral of a matrix valued connections in the cases where the matrices from different points are non-commuting.

where A is the 1-form gauge field and ρ is a representation of the gauge group (so that the push forward ρ_* is a representation of the Lie algebra). For a line C with end points p_i and p_f, under a gauge transformation $A \to gAg^{-1} - ig\mathrm{d}g^{-1}$, the operator $W_\rho(C)$ transformas as:

$$W_\rho(C) \to \rho\left(g(p_f)\right) W_\rho(C) \rho\left(g^{-1}(t_i)\right) , \tag{1.27}$$

so for a closed loop C the operator $W_\rho(C)$ transforms under the adjoint representation of the gauge group and $\mathrm{Tr}\, W_\rho(C)$ is gauge invariant. In supersymmetric theories these operators are invariant under at most half of the supersymmetry or less, depending on the geometry of C and boundary conditions, and in order to preserve any amount of supersymmetry at all the exponential in (1.26) has to be properly modified, which can be done canonically.

In $\mathscr{N} = (2,2)$ GLSM with the gauge group $T = U(1)^k$ there are Wilson lines that are $\mathscr{N} = 2_B$ invariant. First let us consider irreducible representations of T. These representations are parametrized by k-tuples of weights, such as $q := (q^1, \cdots, q^k)$, where q^i is the weight for the i'th $U(1)$ factor of the gauge group. Note that for a representation of $U(1)$, such as $U(1) \ni \lambda \mapsto \lambda^s$, to be truly a representation of $U(1)$ as opposed to that of a multiple cover of it, the weight s has to be an integer.[18] Therefore from now on we will assume that the weights q^i of the $U(1)$ factors of T are all integers, i.e., we assume $q \in \mathbb{Z}^k$. We refer to the irrep of $U(1)^k$ corresponding to the weight q as ρ^q:

$$\rho^q : U(1)^k \to U(\mathbb{C}) \cong U(1) , \qquad \rho^q : g \mapsto \prod_{a=1}^{k} g_a^{q^a} , \tag{1.28}$$

where $g := (g_1, \cdots, g_k)$ is an arbitrary element of $U(1)^k$. For concreteness of formulas, let us imagine a world-sheet with a boundary along the temporal direction. We are taking the y^0 and y^1 coordinates to correspond to time and space respectively and we are considering a boundary at $y^1 = 0$. Now, the Wilson line boundary interaction corresponding to this representation is (c.f. the log of (1.26)):

$$-\frac{i}{2}\int \mathrm{d}y^0 \int \mathrm{d}\theta \mathrm{d}\overline{\theta}\, \rho_*^q(V) = i\int \mathrm{d}y^0\, \rho_*^q(v_0 - \Re\sigma) , \tag{1.29}$$

where V is the vector superfield for $U(1)^k$ and the superspace integral is over the entire $\mathscr{N} = 2_B$ *boundary superspace* [Hor00] making it manifestly 2_B-invariant. A brane characterized by such an elementary boundary interaction will be referred to as:

$$\mathscr{W}(q) . \tag{1.30}$$

More general representations of $U(1)^k$ are characterized by some number, say n, of k-tuple weights $\mathbf{q} := (q_1, \cdots, q_n)$ with $q_i \in \mathbb{Z}^k$, and a multiplicity for each of the weights. Let us denote the module for such a general representation by $\mathscr{V}_{\mathbf{q}}$ and the

[18] A representation $U(1) \ni \lambda \mapsto \lambda^{\frac{a}{b}}$ for some $a, b \in \mathbb{Z}$ with $\gcd(a,b) = 1$, is an $|a|$-fold representation of the $|b|$-fold cover of $U(1)$.

weight spaces for the weights q_i by $\mathscr{V}_{q_i}$. Then this representation can written as:

$$\rho^{\mathbf{q}} : U(1)^k \to U(\mathscr{V}_{\mathbf{q}}) , \qquad \rho^{\mathbf{q}} : g \mapsto \begin{pmatrix} \rho^{q_1}(g)\,\mathrm{id}_{\mathscr{V}_{q_1}} & & \\ & \ddots & \\ & & \rho^{q_n}(g)\,\mathrm{id}_{\mathscr{V}_{q_n}} \end{pmatrix} , \tag{1.31}$$

where ρ^{q_i}'s are the irreps from (1.28). We can interpret these representations by considering the irreducible representations as the basic boundary conditions and introducing $\mathscr{V}_{\mathbf{q}}$ as a Chan-Paton space at the end points of the open strings (see §1.B).[19] The $\mathscr{N} = 2_B$ invariant Wilson line action for a general representation is a direct generalization of (1.29):

$$i \int \mathrm{d}y^0 \, \rho^{\mathbf{q}}_*(v_0 - \Re\sigma) . \tag{1.32}$$

Such a boundary action is a direct sum of elementary boundary actions, and the corresponding notation for a brane described by this boundary action is:

$$\bigoplus_{i=1}^{n} \mathscr{W}(q_i)^{\oplus \dim \mathscr{V}_{q_i}} . \tag{1.33}$$

This notation is limited in its expressibility of all the properties of the brane that we will describe in the rest of this section and for that reason we will not use this notation again until §1.4 where we will be primarily concerned with the gauge charges associated to a brane and this notation will be economic.

1.3.1.2 Fermion number, $\mathbb{Z}_2$: Matrix factorization

The Wilson line action (1.32) that we have discussed so far is $\mathscr{N} = 2_B$ invariant on its own and different choices of such terms correspond to different branes. What *all* B-branes have in common is the minimal boundary action that we must add to the bulk action to preserve $\mathscr{N} = 2_B$ supersymmetry. It turns out [HHP08] that in the presence of nonzero superpotential, in order to preserve 2_B supersymmetry, we have to introduce a composite fermionic field, which we will call Q, living on the boundary of the world-sheet, and Q is a polynomial function of the chiral multiplet fields x^i. The Q-dependent part of the minimal boundary action is [HHP08]:

$$\int \mathrm{d}y^0 \, \mathscr{I}_0 , \qquad \mathscr{I}_0 := -\frac{1}{2}\{Q, Q^\dagger\} + \frac{1}{2}\sum_{i=1}^{N}\left(\psi^i \frac{\partial}{\partial x^i} Q + \mathrm{h.c.}\right) , \tag{1.34}$$

[19] Note that when we mention string we simply mean that our space is one dimensional. In the honest string theory one needs to further couple the world-sheet SCFT to a ghost system to gauge world-sheet diffeomorphism.

where $\psi^i := \psi^i_+ + \psi^i_-$ and x^i are fields from the chiral multiplets restricted to the boundary. The fermionic field Q must satisfy some constraints in order to preserve 2_B supsymmetry which we will discuss shortly. But first note that, as a boundary operator Q acts on the Chan-Paton space $\mathscr{V}$ (just like the boundary operator $\rho_*^{\mathfrak{q}}(v_0 - \Re\sigma)$ in the Wilson line boundary interaction (1.32)). A bit more precisely, since $Q \in \mathbb{C}[x^1, \cdots, x^N] =: \mathbb{C}[x]$, Q acts as an operator on the $\mathbb{C}[x]$-module $\mathscr{V}_x := \mathbb{C}[x] \otimes_{\mathbb{C}} \mathscr{V}$. Since Q is a fermionic operator, it can be assigned a non-trivial charge under a $\mathbb{Z}_2$ group action under which Q^2 is bosonic. This makes the Chan-Paton space $\mathbb{Z}_2$-graded with a graded action of Q:

$$\mathscr{V} = \mathscr{V}^{\mathrm{od}} \oplus \mathscr{V}^{\mathrm{ev}}\,, \qquad \begin{tikzcd} \mathscr{V}_x^{\mathrm{od}} \arrow[r, bend left, "Q"] & \mathscr{V}_x^{\mathrm{ev}} \arrow[l, bend left, "Q"] \end{tikzcd}\,, \tag{1.35}$$

where we have defined $\mathscr{V}_x^{\mathrm{od/ev}} := \mathbb{C}[x] \otimes_{\mathbb{C}} \mathscr{V}^{\mathrm{od/ev}}$. The $\mathbb{Z}_2$ charge measures the fermion number of a vector modulo 2, acting with the fermionic (odd) operator Q changes this charge by 1.

Now on to the constraints on Q. First of all, for the action (1.34) to be gauge invariant, we must impose on Q gauge equivariance:

$$\rho\left(g^{-1}\right) Q(g\cdot x)\rho\left(g\right) = Q(x)\,, \tag{1.36}$$

where ρ is the representation of $U(1)^k$ associated with the Chan-Paton space $\mathscr{V}$, $g \in U(1)^k$, and $x \in \mathbb{C}^N$. The whole purpose of the boundary action (1.34) is to cancel the variation of the bulk superpotential action (world-sheet integral of the last term in (1.3)).[20] To that end, let us note the variation of the superpotential action generated by a 2_B supersymmetry transformation parametrized by the Dirac spinor ε:

$$\delta\,\Re \int_\Sigma \mathrm{d}^2 y \int \mathrm{d}\theta^+ \mathrm{d}\theta^-\, W(\Phi) = -\Re \int \mathrm{d}y^0 \sum_{i=1}^N \overline{\varepsilon}\psi^i \frac{\partial W(x)}{\partial x^i}\,. \tag{1.37}$$

Let us also write down the variation of the boundary interaction from (1.34) generated by ε (note that we are not assuming ε to be constant):[21]

$$\delta\mathscr{I}_0 = \Re\left\{\sum_{i=1}^N \left(\overline{\varepsilon}\psi^i \frac{\partial Q^2(x)}{\partial x^i}\right) - \left[\overline{\varepsilon}Q^\dagger, Q^2\right]\right\} - i\mathscr{D}_0\left(\overline{\varepsilon}Q + \varepsilon Q^\dagger\right) + i\left(\dot{\overline{\varepsilon}}Q + \dot{\varepsilon}Q^\dagger\right)\,, \tag{1.38}$$

[20] The rest of the minimal boundary action to cancel the variation of the world-sheet integral of the rest of the terms in (1.3) is not going to play any part in our discussion, so we are not going to talk about them. They can be found in [Hor00, HKLM02, HHP08].

[21] The actual 2_B supersymmetry is generated by constant ε, so in a variation generated by time dependent ε, any term that behaves as $\mathscr{O}(\dot{\varepsilon})$ can be ignored as far as symmetry preservation is concerned, but such terms help to recognize various contributions to the Noether charge associated to the symmetry.

where $\mathscr{D}_0$ is the gauge covariant derivative along the y^0 direction. The boundary integral of the first term in (1.38) cancels (1.37) if and only if Q satisfies:

$$Q^2(x) = W(x)\,\mathrm{id}_{\mathscr{V}}\,. \tag{1.39}$$

The above equation is referred to as a *matrix factorization* of W. The second term in (1.38) is a total derivative and does not contribute when integrated over the boundary. The third term will vanish when we make the supersymmetry parameter ε constant, but it shows, via Noether's construction, that the supercharge generating $\mathscr{N}=2_B$ supersymmetry gets modified at the boundary:

$$\boldsymbol{Q}_{\text{bulk}} \to \boldsymbol{Q}_B := \boldsymbol{Q}_{\text{bulk}} + \boldsymbol{Q}_{\text{bdry}}\,, \qquad \boldsymbol{Q}_{\text{bdry}} = iQ\big|_{y^1=0} - iQ\big|_{y^1=L}\,, \tag{1.40}$$

where we are considering two temporal boundaries at spatial positions $y^1=0$ and $y^1=L$. Importantly, this boundary modifiation ensures the nilpotence of the total supercharge. The bulk supercharge, in the presence of a non-trivial superpotential and the boundary, is not nilpotent [HHP08]:

$$\boldsymbol{Q}_{\text{bulk}}^2 = W\big|_{y^1=L} - W\big|_{y^1=0}\,, \tag{1.41}$$

and this is canceled by $\boldsymbol{Q}_{\text{bdry}}^2$ due to (1.39) so that we end up with the $\mathscr{N}=2_B$ algebra (see (1.87)):

$$\boldsymbol{Q}_B^2 = 0\,. \tag{1.42}$$

Equations (1.36) and (1.39) are all the constraints that Q has to satisfy.

1.3.1.3 Vector R-symmery, $U(1)_V$: Differential grading

We are interested in GLSMs with vector $U(1)$ R-symmetry, denoted by $U(1)_V$, so that the theory flows to a non-trivial conformal fixed point in the extreme infrared. In that case, the bulk supercharge $\boldsymbol{Q}_{\text{bulk}}$ has $U(1)_V$-charge 1 (1.88). Since the boundary odd operator Q is now a part of the total supercharge along with $\boldsymbol{Q}_{\text{bulk}}$ (see (1.40)), we must assign the same $U(1)_V$-charge to Q. Consequently, the Chan-Paton space $\mathscr{V}$ must carry an action of $U(1)_V$. We pick a representation $R: U(1)_V \to U(\mathscr{V})$ with weights $(v_{\min}, v_{\min}+1, \cdots, v_{\max})$ where $v_j \in \mathbb{Z}$. We require the representation to be such that the R-charge modulo 2 matches with the $\mathbb{Z}_2$ grading of $\mathscr{V}$, i.e., a $U(1)_V$-weight space decomposition of $\mathscr{V}$ satisfies:

$$\mathscr{V} = \bigoplus_{j=v_{\min}}^{v_{\max}} \mathscr{V}^j \qquad \text{with} \qquad \mathscr{V}^{\text{od}} = \bigoplus_{j:\text{odd}} \mathscr{V}^j\,, \quad \mathscr{V}^{\text{ev}} = \bigoplus_{j:\text{even}} \mathscr{V}^j\,, \tag{1.43}$$

where $\mathscr{V}^j$ is the $U(1)_V$-weight space with weight j so that the representation R can be explicitly written as:

$$R:\lambda\mapsto\begin{pmatrix}\lambda^{v_{\min}}\,\mathrm{id}_{\mathscr{V}^{v_{\min}}} & & \\ & \ddots & \\ & & \lambda^{v_{\max}}\,\mathrm{id}_{\mathscr{V}^{v_{\max}}}\end{pmatrix},\qquad \lambda\in U(1)_V\,. \tag{1.44}$$

The R-symmetry commutes with the gauge symmetry, implying that each $\mathscr{V}^j$ is a direct sum of $U(1)^k$-weight spaces. Since Q increases the R-charge of a vector by 1, Q can be thought of as a collection of odd operators $Q=(\mathrm{d}_{v_{\min}},\cdots,\mathrm{d}_{v_{\max}})$:

$$\cdots\xrightarrow{d_{j-1}}\mathscr{V}_x^j\xrightarrow{d_j}\mathscr{V}_x^{j+1}\xrightarrow{d_{j+1}}\cdots\,. \tag{1.45}$$

With a nonzero superpotential this is not a standard cochain complex since the differential Q is not nilpotent: $\mathrm{d}_{j+1}\circ\mathrm{d}_j=W\,\mathrm{id}_{\mathscr{V}_x^j}$ (due to (1.39)). This is sometimes referred to as a differential *twisted* by W. This can be remedied by defining the ring:

$$\mathscr{R}_W:=\mathbb{C}[x^1,\cdots,x^n]/\langle W\rangle\,, \tag{1.46}$$

where $\langle W\rangle$ is the ideal generated by the polynomial W, and instead of working with the $\mathbb{C}[x]$-module $\mathscr{V}_x$, we now consider the following $\mathscr{R}_W$-module:

$$\mathscr{V}_W:=\mathscr{R}_W\otimes_{\mathbb{C}}\mathscr{V}\,. \tag{1.47}$$

The good thing about this module is that W acts trivially on it, furthermore $\mathscr{V}_W$ inherits all the gradings of $\mathscr{V}$.[22] Therefore, we can build a genuine cochain complex using $\mathscr{V}_W$:

$$\cdots\xrightarrow{\mathrm{d}_{j-1}}\mathscr{V}_W^j\xrightarrow{\mathrm{d}_j}\mathscr{V}_W^{j+1}\xrightarrow{\mathrm{d}_{j+1}}\cdots\,. \tag{1.48}$$

Let us summerize our description of a GLSM B-brane:

A GLSM B-brane $\mathscr{B}$ is defined by the data $(\mathscr{V},\rho,R,Q)$, where $\mathscr{V}$ is a Chan-Paton space carrying a representation ρ of the gauge group $T=U(1)^k$, a representation R of the $U(1)_V$ R-symmetry group that commutes with T, and a matrix factorization $Q^2=W\,\mathrm{id}_{\mathscr{V}}$ of the superpotential W, where the T-equivariant operator Q has R-charge 1. All these data is encoded in the complex (1.48) of the *twisted* modules $\mathscr{V}_W^j=(\mathbb{C}[x]/\langle W\rangle)\otimes_{\mathbb{C}}\mathscr{V}^j$ where j refers to the $\mathbb{Z}$-grading by $U(1)_V$, i.e., $\mathscr{V}^j$ is the $U(1)_V$-weight space with weight j and the implicit $\mathbb{Z}^k$-gradings by T are compatible with this $\mathbb{Z}$-grading.

We will refer to the complex (1.48) associated to a B-brane $\mathscr{B}$ as $\mathscr{C}(\mathscr{B})$. In the next section we will relate the $\boldsymbol{Q}_B$-invariant open string states to certain morphisms between these cochain complexes.

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[22] We use the same notation to denote the gradings of $\mathscr{V}_W$ as we did for $\mathscr{V}$, just by replacing $\mathscr{V}$ with $\mathscr{V}_W$.

1.3.2 Open string states in the chiral sector

A satisfactory description of branes must provide two aspects: how to characterize the branes, and how to characterize the states in the Hilbert space that we get after quantizing a string[23] in the presence of these branes. By now we have some idea of how to do the former and in this section we discuss the latter in the limited context of BPS or chiral states.

1.3.2.1 Some generalities and the chiral ring

The chiral sector of the open string Hilbert space consists of states that are invariant under $\boldsymbol{Q}_B$:

$$\boldsymbol{Q}_B|\Psi\rangle = 0\,. \tag{1.49}$$

This sector, owing to being $\boldsymbol{Q}_B$-invariant, is robust under various supersymmetry preserving deformations, in particular, it is robust under rescaling of the world-sheet metric,[24] i.e., renormalization. Therefore, the chiral states are particularly interesting if we care about the low energy dynamics of our theory. Analogous to (1.49), chiral operators are the $\boldsymbol{Q}_B$-invariant operators:

$$[\boldsymbol{Q}_B, \mathscr{O}] = 0\,. \tag{1.50}$$

The hermitian conjugates of (1.49) and the above equation define the *anti-chiral* states and operators respectively. There is in fact a one to one correspondence between chiral states and local chiral operators. This follows from the fact that the chiral states and operators are invariant under RG flow and therefore the state-operator correspondence that exists between them in the IR CFT works in the UV GLSM as well. In this review we are interested in chiral operators that can be inserted *on* the boundary. Once inserted, we can have two different $\mathscr{N} = 2_B$ boundary conditions on the two sides of this operator, and a path integral on a half-disc in a half-plane with such boundary conditions gives us the chiral state corresponding to this operator. This is essentially the state-operator correspondence between a chiral open string state between two boundaries of an infinite strip and a local chiral operator on the boundary of a half-plane with two boundary conditions on the two sides (see figure 1.2).

These chiral operators have non-singular operator product expansion among them [LVW89] and this allows to define the following product of two chiral operators $\mathscr{O}_1$ and $\mathscr{O}_2$:

$$(\mathscr{O}_1\mathscr{O}_2)(y) := \lim_{y'\to y} \mathscr{O}_1(y)\mathscr{O}_2(y')\,. \tag{1.51}$$

[23] A time slice of the worldsheet.

[24] The chiral sector is invariant under any continuous world-sheet metric deformation.

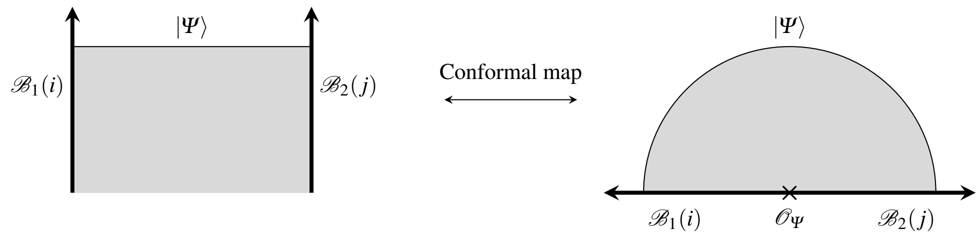

Fig. 1.2 State-operator correspondence: The operator $\mathscr{O}_\Psi$ on the right hand side is defined so that a path integral on the shaded region in the upper half-plane with the boundary conditions $\mathscr{B}_1(i)$ and $\mathscr{B}_2(j)$ produces the wave function for the state $|\Psi\rangle \in \mathscr{H}(\mathscr{B}_1(i), \mathscr{B}_2(j))$ (c.f. (1.89)).

The product of two chiral operators is again a chiral operator and this turns the chiral sector into a ring called the *chiral ring*.[25] If we denote the chiral ring between two branes[26] $\mathscr{B}_i$ and $\mathscr{B}_j$ as $\mathscr{H}_{\boldsymbol{Q}_B}(\mathscr{B}_i, \mathscr{B}_j)$ then the OPE defines a map as follows:

$$\mathscr{H}_{\boldsymbol{Q}_B}(\mathscr{B}_1, \mathscr{B}_2) \times \mathscr{H}_{\boldsymbol{Q}_B}(\mathscr{B}_2, \mathscr{B}_3) \to \mathscr{H}_{\boldsymbol{Q}_B}(\mathscr{B}_1, \mathscr{B}_3)\,. \tag{1.52}$$

In constructing the chiral ring we are not only restricting to operators obeying (1.50), we also identify $\boldsymbol{Q}_B$-exact operators[27] with zero because all correlation functions involving a $\boldsymbol{Q}_B$-exact operator and other chiral operators are zero:

$$\langle [\boldsymbol{Q}_B, \mathscr{O}_1] \mathscr{O}_2 \rangle = 0\,, \tag{1.53}$$

which follows from a Ward identity[28] for $\boldsymbol{Q}_B$ and the chirality of $\mathscr{O}_2$. Therefore, the chiral ring is really the $\boldsymbol{Q}_B$-cohomology of the local operators.

1.3.2.2 The chiral ring between two branes

The local operators of the theory are arbitrary functions of the fields. In the presence of boundary, in order to formulate a well posed initial value problem, the dynamical fields need to be constrained by boundary conditions (such as Dirichlet, Neuman or generalizations thereof). These boundary conditions can break some symmetries of the bulk theory, we of course want to preserve $\mathscr{N} = 2_B$ supersymmetry. Before mentioning the 2_B-invariant boundary conditions, let us take a simplifying limit. We will construct the chiral boundary local operators in the far ultraviolet, which

[25] The chiral ring is also an aglebra over $\mathbb{C}$, but it is more commonly referred to as a ring.

[26] In the language of §1.B, each brane $\mathscr{B}_i$ comes with an index set $\mathscr{I}_i$ of the Chan-Paton indices, so that for each index $k \in \mathscr{I}_i$ we have a basic boundary condition $\mathscr{B}_i(k)$ associated to the brane $\mathscr{B}_i$, and the Hilbert space for the open string between $\mathscr{B}_1$ and $\mathscr{B}_2$ has the following decomposition: $\mathscr{H}(\mathscr{B}_1, \mathscr{B}_2) = \bigoplus_{\substack{i \in \mathscr{I}_1 \\ j \in \mathscr{I}_2}} \mathscr{H}(\mathscr{B}_1(i), \mathscr{B}_2(j))$.

[27] Which are automatically chiral due to the nilpotence of $\boldsymbol{Q}_B$.

[28] Which simply says that the expectation value of the supersymmetry variation of an operator in a supersymmetric vacuum is zero.

corresponds to the limit $e \to 0$. Even though the chiral ring is RG invariant, there are some subtleties involved with this limit, which we are going to ignore in this review, for details see [HHP08]. In this limit, the coefficient in front of the vector multiplet kinetic terms (see (1.3)) diverges and those fields freeze to their classical values, becoming backgrounds. So we only have the chiral multiplet fields to construct the chiral ring with. In the limit $e \to 0$, the standard $\mathscr{N} = 2_B$ invariant boundary conditions are Neuman:

$$\partial_1 x^i = 0\,, \qquad \psi^i_+ = \psi^i_-\,, \qquad \partial_1(\psi^i_+ + \psi^i_-) = 0\,, \qquad F^i = 0\,. \tag{1.54}$$

We will make the further assumption that our strings are sufficiently small compared to the length scale of the target space. This allows us to treat the string as point particle and ignore stringy excitations. Since we are assuming the target space volume to be too large compared to the string length, this approximation is called the *large volume limit*. It is also known as *zero mode approximation* becuase, since we are ignoring the stringy excitations, we will only use the zero modes of the fields to construct our operators. It turns out though, that this approximation is exact for the chiral ring .[29] This can be argued by showing that the chiral ring is independent of any length scale on the world-sheet. With these approximations, the complex computing $\boldsymbol{Q}_B$-cohomology of the *bulk* chiral operators can be constructed just from the zero modes of x^i, $\overline{x}^i$, and $\overline{\psi}^i_\pm$ [Wit91].[30] When we put these operators on the boundary, the boundary conditions (1.54) further reduce the set of fields by setting $\overline{\psi}^i_+ - \overline{\psi}^i_-$ to zero. The action of the bulk supercharge $\boldsymbol{Q}_{\text{bulk}}$ on the fields we have left is as follows:[31]

$$[i\boldsymbol{Q}_{\text{bulk}}, x^i] = 0\,, \qquad [i\boldsymbol{Q}_{\text{bulk}}, \overline{x}^i] = \overline{\psi}^i_+ + \overline{\psi}^i_-\,, \qquad \{i\boldsymbol{Q}_{\text{bulk}}, \overline{\psi}^i_+ + \overline{\psi}^i_-\} = 0\,. \tag{1.55}$$

If we identify $\overline{\psi}^i_+ + \overline{\psi}^i_-$ with $\mathrm{d}\overline{x}^i$, then the action of $\boldsymbol{Q}_{\text{bulk}}$ coincides with the action of the Dolbeault differential $\overline{\partial}$. The action of the full superchage $\boldsymbol{Q}_B$ will also include the action of the boundary supercharge $\boldsymbol{Q}_{\text{bdry}}$ which we will discuss later.

The operators we can construct from $x^i, \overline{x}^i$ and $\mathrm{d}\overline{x}^i$ correspond to anti-holomorphic forms in $\Omega^{0,\bullet}\left(\mathbb{C}^N\right)$ and to get the chiral ring we take the $\boldsymbol{Q}_B$-cohomology. Let us clarify all the data we have. We are interested in chiral operators interpolating between two branes, let us say $\mathscr{B}_1 = (\mathscr{V}_1, \rho_1, R_1, Q_1)$ and $\mathscr{B}_2 = (\mathscr{V}_2, \rho_2, R_2, Q_2)$. Then according to the general discussion of §1.B, the operators should be valued in $\mathrm{Hom}(\mathscr{V}_1, \mathscr{V}_2)$. The $\mathrm{Hom}(\mathscr{V}_1, \mathscr{V}_2)$-valued anti-holomorphic forms are charged under both the gauge group and the $U(1)_V$ R-symmetry group. The R-charge is relevant for the differential grading since the supercharge $\boldsymbol{Q}_B$ has R-charge 1. So we will

[29] This is a nontrivial result, for example, it is not true for the twisted chiral ring ($\boldsymbol{Q}_A$-cohomology) which receives world-sheet instanton corrections .

[30] Let us say $\psi^i_+\mathscr{O}$ is chiral for some operator $\mathscr{O}$. Then $0 = [\boldsymbol{Q}_{\text{bulk}}, \psi^i_+\mathscr{O}] = -2(\partial_+ x^i)\mathscr{O} - \psi^i_+[\boldsymbol{Q}_{\text{bulk}}, \mathscr{O}]$ where $2\partial_+ = \partial_0 + \partial_1$. If $\mathscr{O}$ is chiral then $2(\partial_+ x^i)\mathscr{O} = 0$ implies $\mathscr{O}$ is zero and so is $\psi^i_+\mathscr{O}$. If $\mathscr{O}$ is not chiral then $2(\partial_+ x^i)\mathscr{O} + \psi^i_+[\boldsymbol{Q}_{\text{bulk}}, \mathscr{O}] = 0$ implies $\mathscr{O}$ must be of the form $\psi^i_+\mathscr{O}'$ for some $\mathscr{O}'$ in which case $\psi^i_+\mathscr{O} = \psi^i_+\psi^i_+\mathscr{O}' = 0$ since ψ^i_+ is anti-commuting.

[31] The action of $\mathscr{N} = (2,2)$ supersymmetry on the fields of GLSM can be found in [Wit93, HHP08].

write $\mathrm{Hom}^\bullet(\mathscr{V}_1,\mathscr{V}_2)$ as a reminder that these maps carry $\mathbb{Z}$-grading by $U(1)_V$. Furthermore, only gauge invariant operators are physical. Now the chiral ring elements with R-charge p are given by the $\boldsymbol{Q}_B$-cohomology:

$$\mathscr{H}^p_{\boldsymbol{Q}_B}(\mathscr{B}_1,\mathscr{B}_2) = H^p_{\boldsymbol{Q}_B}\left(\Omega^{0,\bullet}\left(\mathbb{C}^N, \mathrm{Hom}^\bullet(\mathscr{V}_1,\mathscr{V}_2)\right)^T\right), \tag{1.56}$$

where the degree of the complex is a sum of the form degree and the degree of the morphisms in $\mathrm{Hom}^\bullet(\mathscr{V}_1,\mathscr{V}_2)$.

The action of $\boldsymbol{Q}_B$ on an operator $\mathscr{O}$ on the boundary between $\mathscr{B}_1$ and $\mathscr{B}_2$ is given by:

$$i\boldsymbol{Q}_B : \mathscr{O} \mapsto \overline{\partial}\mathscr{O} + Q_2\mathscr{O} - (-1)^{|\mathscr{O}|}\mathscr{O}Q_1\,, \tag{1.57}$$

where $|\mathscr{O}|$ is the R-charge[32] of $\mathscr{O}$ and the action of the boundary supercharges comes from Noether's construction of the supercharge in presence of the boundaries. Using the property of $\mathbb{C}^N$ that any $\overline{\partial}$-closed p-form on $\mathbb{C}^N$ is $\overline{\partial}$-exact for $p>0$, it can be shown that the cohomologies (1.56) are concentrated on *holomorphic or polynomial functions* [HHP08], i.e.:

$$\mathscr{H}^p_{\boldsymbol{Q}_B}(\mathscr{B}_1,\mathscr{B}_2) \cong H^p_{\boldsymbol{Q}_B^{\mathrm{hol}}}\left(\mathrm{Hom}^\bullet(\mathscr{V}_1,\mathscr{V}_2)^T\right). \tag{1.58}$$

Restricted to the subspace of holomorphic functions, the action of the supercharge (1.57) simplifies to

$$i\boldsymbol{Q}_B : \mathscr{O} \mapsto Q_2\mathscr{O} - (-1)^{|\mathscr{O}|}\mathscr{O}Q_1\,. \tag{1.59}$$

Now recall from the end of §1.3.1 that the data of a B-brane $\mathscr{B}$ can be encoded in a cochain complex that we called $\mathscr{C}(\mathscr{B})$ (1.48). These complexes were built out of $\mathscr{R}_W$-modules $\mathscr{V}_W$ (see (1.47) and (1.46)) which were $\mathbb{Z}^k$-graded (by the gauge group $T=U(1)^k$).[33] Gauge invariant chiral holmorphic functions valued in $\mathrm{Hom}(\mathscr{V}_1,\mathscr{V}_2)$ now precisely correspond to chain maps[34] betwen $\mathscr{C}(\mathscr{B}_1)$ and $\mathscr{C}(\mathscr{B}_2)$ as graded $\mathscr{R}_W$-modules.[35] Identifying a chain map of the form $[\boldsymbol{Q}_B,\mathscr{O}]$ with zero for some arbitrary local operator $\mathscr{O}$, as we do in the chiral ring, is then equivalent to identifying chain maps upto homotopy . Thus we reach the conclusion that the chiral ring elements between two branes $\mathscr{B}_1$ and $\mathscr{B}_2$ are in one to one correspondence with the cochain maps upto homotopy, between $\mathscr{C}(\mathscr{B}_1)$ and $\mathscr{C}(\mathscr{B}_2)$ where these are $\mathbb{Z}^k$-graded cochain complexes of $\mathscr{R}_W$-modules. One last thing to add is that, a cochain map will have differential degree, i.e. R-charge, zero, on the other hand the chiral ring contains elements of all possible degrees ($\mathbb{Z}$-graded). To allow this, given two branes $\mathscr{B}_1$ and $\mathscr{B}_2$ we must consider cochain maps between $\mathscr{C}(\mathscr{B}_1)$ and

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32 Which is a sum of the form degree (as $\overline{\psi}^i_+ + \overline{\psi}^i_-$ has R-charge 1) and the degree of the $\mathrm{Hom}(\mathscr{V}_1,\mathscr{V}_1)$ part (which is due to the x^i's and $\overline{x}^i$'s).

33 In addition to the (differential) $\mathbb{Z}$-grading by $U(1)_V$.

34 The condition of chirality $\boldsymbol{Q}_B(\mathscr{O}) = [\boldsymbol{Q}_B,\mathscr{O}] = 0$ for $\mathrm{Hom}(\mathscr{V}_1,\mathscr{V}_2)$-valued holomorphic functions is equivalent to the condition for $\mathscr{O} : \mathscr{C}(\mathscr{B}_1) \to \mathscr{C}(\mathscr{B}_2)$ to be a cochain map (see (1.59)).

35 A morphism between two $\mathbb{Z}^k$-graded $\mathscr{R}_W$-modules is a morphism that preserves the grading, i.e., the morphisms themselves have $\mathbb{Z}^k$-degree zero, in our context this means that the morphisms are gauge invariant.

$\mathscr{C}(\mathscr{B}_2)[p]$ where $[p]$ denotes the same complex shifted p times to the left. Finally, our description of the chiral ring is:

The degree p chiral ring elements between two branes $\mathscr{B}_1$ and $\mathscr{B}_2$ are the morphisms between $\mathscr{C}(\mathscr{B}_1)$ and $\mathscr{C}(\mathscr{B}_2)[p]$ in the homotopy category of $\mathbb{Z}^k$-graded complexes of $\mathscr{R}_W$-modules:

$$\mathscr{H}^p_{\boldsymbol{Q}_B}(\mathscr{B}_1,\mathscr{B}_1) \cong \mathrm{Hom}_{\mathbf{K}(\text{gr-}\mathscr{R}_W)}(\mathscr{C}(\mathscr{B}_1),\mathscr{C}(\mathscr{B}_2)[p]) \ . \tag{1.60}$$

From the discussion of this section and the last, it is clear that, as far as the chiral sector is concerned, the GLSM B-branes are well described by the homotopy category of some graded modules. The branes appear as objects of this category and the chiral ring between them corresponds to the morphisms.

1.4 Relations with the IR branes

In this section we will define a projection from the GLSM B-branes to the IR B-branes and we will see that we get the *stable* IR branes if we identify the projections up to quasi-isomorphisms. This tells us that the low energy branes are objects in a derived category. We will only sketch out the key ideas here, and to avoid many technical complexities we will describe the relevant ideas for theories *without* a superpotential, so that at a generic point in the Kähler moduli space $\mathscr{M}_K$, the low energy theory is described by a non-linear sigma model on a non-compact toric variety X_r (1.17).

1.4.1 Tracking branes to the infrared

The gauge theory becomes strongly coupled as we go to the infrared, the deep infrared limit corresponds to taking $e \to \infty$. In this limit the kinetic terms for the vector multiplet fields vanish (see (1.3)) and therefore these fields acquire algebraic equations of motion. Let us illustrate this limit for a $U(1)$ gauge field with a Wilson line interaction on a boundary with charge q. Now, in the limit $e \to \infty$, the zeroth component v_0 of the gauge field appears in the action as:

$$\frac{1}{2\pi}\int_\Sigma \mathrm{d}^2y \sum_{i=1}^N Q_i^2|x^i|^2 \left(v_0^2 - \frac{\sum_{j=1}^N iQ_j(\bar{x}^j\partial_0 x^j - \partial_0\bar{x}^j x^j)}{\sum_{i=1}^N Q_i^2|x^i|^2} v_0 + \cdots \right) - \int_{\partial\Sigma}\left(\frac{\theta}{2\pi}+q\right)v_0\,, \tag{1.61}$$

where the $\cdots$ includes terms involving fermions and will vanish on a vacuum background. We can do the same computation for v_1 and after putting them together, the algebraic equation of motion for the one form v that we get by varying these expressions is that v becomes the pullback:

$$v = x^* A\,, \tag{1.62}$$

where A is the connection of the holomorphic line bundle $\mathscr{O}(1)$ on X_r. Then the Wilson line (1.29) with charge q corresponds to the holomorphic line bundle $\mathscr{O}(q)$, which is the low energy brane supported on X_r corresponding to the Wilson line brane in the GLSM. We thus get the following projection from GLSM B-branes to IR B-branes:

$$\pi : \mathscr{W}(q) \mapsto \mathscr{O}(q)\,. \tag{1.63}$$

Not all branes are stable in the infrared however. We have to find a zero of the potential $\{Q, Q^\dagger\}$ in the boundary action (1.34), just as we found the zero of the potential of the bulk action in §1.2.2. If $\{Q, Q^\dagger\}$ is positive everywhere on X_r then the entire brane is unstable and it will vanish by the mechanism of brane anti-brane annihilation [Sen98]. Such a brane is called *trivial* in the infrared. For example, if we do not care about the $U(1)_V$ R-symmetry then we can have a constant superpotential $W = c$ and then the ring $\mathscr{R}_W$ (1.46) and consequently the modules $\mathscr{V}_W$ (1.47) will vanish, resulting in a null complex (brane) (1.48). More interesting examples of infrared trivial branes arise from the existence of the deleted sets Δ_r while constructing X_r as a quotient (1.18). If $\{Q, Q^\dagger\}$ is strictly positive everywhere in $\mathbb{C}^N \backslash \Delta_r$, then the corresponding brane will be infrared trivial. The fact that the deleted set varies from phase to phase now makes phase transition for branes rather interesting, since a UV brane that is trivial in one IR phase may be non-trivial in another. This makes the projection of a GLSM B-brane complex $\mathscr{C}(\mathscr{B})$ to an IR complex phase dependent, let us make it explicit in notation:

$$\pi_r : \mathscr{W}(q) \mapsto \mathscr{O}(q)\,, \qquad \pi_r : Q(x)\big|_{x \in \mathbb{C}^N} \mapsto Q(x)\big|_{x \in X_r}\,. \tag{1.64}$$

Similarly, for any cochain map $\phi : \mathscr{C}(\mathscr{B}_1) \to \mathscr{C}(\mathscr{B}_2)$ between two GLSM B-brane complexes, the projection $\pi_r(\phi) : \pi_r(\mathscr{C}(\mathscr{B}_1)) \to \pi_r(\mathscr{C}(\mathscr{B}_2))$ is defined by projecting $x \in \mathbb{C}^N$ to $x \in X_r$:

$$\pi_r : \phi(x)\big|_{x \in \mathbb{C}^N} \mapsto \phi(x)\big|_{x \in X_r}\,. \tag{1.65}$$

Given two UV B-branes we would like to know whether they will flow to the same IR branes or not. The non-trivial answer to this question is that:[36]

Two GLSM B-branes $\mathscr{B}_1$ and $\mathscr{B}_2$ flow to the same IR brane in a phase containing $r \in \mathscr{M}_K$, if there exists a *quasi-isomorphism* $\varphi : \pi_r(\mathscr{C}(\mathscr{B}_1)) \to \pi_r(\mathscr{C}(\mathscr{B}_2))$.

[36] A quasi-isomorphism is a cochain map that induces an isomorphism of cohomology. Recall that in physical terms, cochain maps are chiral ring elements.

We will omit the proof of this here, which can be found in [HHP08]. The key observations behind this proof are that, a brane can be deformed to another quasi-isomorphic brane by brane anti-brane annihilation and some deformations of the boundary actions that do not affect the low energy physics. We can observe that if two branes have quasi-isomorphic projections in some phase then we can turn one of them into an anti-brane and the combined brane anti-brane system is indeed trivial in the infrared (in that phase), signaling the fact that the two original branes are indistinguishable at low energy in that phase. Concretely, turning one of the branes, say $\mathscr{B}_1$, into anti-brane entails shifting the associated complex $\mathscr{C}(\mathscr{B}_1)$ by a step, i.e., we consider the complex $\mathscr{C}(\mathscr{B}_1)[1]$. Then the bound state of the two branes is the *mapping cone* of the respective complexes, which depends on the choice of a cochain map $\phi : \mathscr{C}(\mathscr{B}_1) \to \mathscr{C}(\mathscr{B}_2)$. We denote the cone as $C(\phi)$:

$$C(\phi) = \mathscr{C}(\mathscr{B}_1)[1] \oplus \mathscr{C}(\mathscr{B}_2)\,, \tag{1.66}$$

with the differential:

$$Q_{C(\phi)} := \begin{pmatrix} -Q_1 & 0 \\ \phi & Q_2 \end{pmatrix}. \tag{1.67}$$

It is not difficult to prove that the complex $\pi_r(C(\phi))$ is exact if and only if $\pi_r(\phi)$ is a quasi-isomorphism. An exact complex is quasi-isomorphic to the null complex by a trivial cochain map: $\pi_r(C(\phi)) \to 0$. Therefore we see that complete brane anti-brane annihilation takes place in the brane $\mathscr{C}(\mathscr{B}_1)[1] \oplus \mathscr{C}(\mathscr{B}_2)$ if and only if $\pi_r(\mathscr{B}_1)$ and $\pi_r(\mathscr{B}_2)$ are quasi-isomorphic.[37] We thus reach the conclusion that the low energy branes are better described as objects in a derived version of the homotopy category that describes the GLSM B-branes.[38]

Let us denote the homotopy category of GLSM B-branes (in a theory with N chiral multiplets and gauge group T) as $\mathfrak{D}(\mathbb{C}^N, T)$ and the derived category of IR B-branes on the toric variety X_r as $\mathrm{D}(X_r)$.[39] Then the projection (1.64) is a map:

$$\pi_r : \mathfrak{D}(\mathbb{C}^N, T) \to \mathrm{D}(X_r)\,. \tag{1.68}$$

Note that now we are thinking of this map as first taking the projection (1.64) and then *localizing* the quasi-isomorphisms. Localizing simply means adding formal inverses of the quasi-isomorphisms together with all the compositions involving these inverses. Then we can treat the quasi-isomorphisms as honest isomorphisms

[37] Perhaps we should say "if $\pi_r(\mathscr{C}(\mathscr{B}_1))$ and $\pi_r(\mathscr{C}(\mathscr{B}_2))$ are quasi-isomorphic," but we will not put much effort into distinguishing a brane from its complex.

[38] In passing from a homotopy category to the derived category one introduces formal inverses for the quasi-isomorphisms (and all compositions of morphisms involving these inverses) so that they can be treated as genuine isomorphisms. Also note that, this conclusion is consistent with the, by now standard, result that the low energy branes in a geometric phase are objects in the derived category of coherent sheaves supported on the CY target space of the IR sigma model [Kon94, Sha99, Dou01].

[39] If we have non-trivial superpotential we need to mention that as well in the notation for the GLSM B-brane category.

and the isomorphism classes of the objects are now in one to one correspondence with the stable IR B-branes.

Example 2. Let us look at a concrete example of quasi-isomorphism in the simple $U(1)$ gauge theory that we considered in example 1. We consider the GLSM B-brane $\mathscr{B}$ in the phase $r > 0$:[40]

$$\mathscr{C}(\mathscr{B}): \quad \mathscr{W}^{-3}(-1) \xrightarrow{d_{-3}} \mathscr{W}^{-2}(0)^{\oplus 3} \xrightarrow{d_{-2}} \mathscr{W}^{-1}(1)^{\oplus 3} \xrightarrow{d_{-1}} \mathscr{W}^{0}(2)\,, \tag{1.69}$$

where the differentials are defined as:

$$d_{-3} := \begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix}, \qquad d_{-2} := \begin{pmatrix} 0 & x^3 & -x^2 \\ -x^3 & 0 & x^1 \\ x^2 & -x^1 & 0 \end{pmatrix}, \qquad d_{-1} := \begin{pmatrix} x^1 & x^2 & x^3 \end{pmatrix}. \tag{1.70}$$

The differential on the total complex $\mathscr{C}(\mathscr{B})$ is:

$$Q = \begin{pmatrix} 0 & & & \\ d_{-3} & & & \\ & d_{-2} & & \\ & & d_{-1} & 0 \end{pmatrix}, \quad \text{satisfying} \quad \{Q, Q^\dagger\} = \left(\sum_{i=1}^{3} |x^i|^2\right) \mathrm{id}_{\mathscr{C}(\mathscr{B})}\,. \tag{1.71}$$

Recalling that the deleted set in the geometric phase $r > 0$ was $\Delta_+ = \{x^1 = x^2 = x^3 = 0\}$, we see that the boundary potential $\{Q, Q^\dagger\}$ is strictly positive in this phase and therefore the brane $\mathscr{B}$ is trivial in the IR, i.e. $\pi_+(\mathscr{B}) \cong 0$. It can also be checked that the projection of the complex (1.69) in this phase is exact, reaching the same conclusion that it is quasi-isomorphic to the null complex. This implies that we can write this brane as a bound state of two branes $\mathscr{C}(\mathscr{B}_1)[1]$ and $\mathscr{C}(\mathscr{B}_2)$ so that $\pi_+(\mathscr{B}_1)$ and $\pi_+(\mathscr{B}_2)$ are quasi-isomorphic. This can be achieved by breaking the complex at an arrow and shifting one of the two resulting complexes, e.g.:

$$\begin{array}{ll} \mathscr{C}(\mathscr{B}_1): & \mathscr{W}^{-2}(-1) \xrightarrow{d_{-3}} \mathscr{W}^{-1}(0)^{\oplus 3} \xrightarrow{d_{-2}} \mathscr{W}^{0}(1)^{\oplus 3} \\ & \qquad\qquad\qquad\qquad\qquad\qquad\qquad \Big\downarrow{\scriptstyle d_{-1}} \\ \mathscr{C}(\mathscr{B}_2): & \qquad\qquad\qquad\qquad\qquad\qquad\qquad \mathscr{W}^{0}(2) \end{array} \tag{1.72}$$

The cochain map $\pi_+(d_{-1}) : \pi_+(\mathscr{C}(\mathscr{B}_1)) \to \pi_+(\mathscr{C}(\mathscr{B}_2))$ being the quasi-isomorphism.

On the other hand, in the orbifold phase $r < 0$, the deleted set is $\Delta_- = \{x^4 = 0\}$ and the boundary potential is vanishing at the orbifold point $p := \{x^1 = x^2 = x^3 = 0\}$. Therefore, the brane $\mathscr{B}$ localizes at p at low energy and, in particular, is non-

[40] The following complex represents the cochain complex (1.48), which in the absence of a superpotential is the same as (1.45). We are using the notation $\mathscr{W}$ introduced in the context of Wilson line branes (see "Wilson line branes" in §1.3.1) instead of using the Chan-Paton spaces to highlight the gauge charges, which plays a crucial role in the computations of the next section. The superscript on $\mathscr{W}$ denotes the R-charge.

vanishing. This also implies that $\pi_-(\mathscr{B}_1)$ and $\pi_-(\mathscr{B}_2)$ are *not* quasi-isomorphic in this phase.

1.4.2 Grade restriction rule

The next step in the story of branes is to consider how to transport the IR B-branes from one phase to another. The GLSM approach to brane proves to be particularly suited to answer this question. The difficulties in transporting branes arise when we consider branes near a phase boundary. To precisely study the nature of the branes near such singular loci, we have to be careful about quantum corrections. We will proceed analogously as we did in computing quantum corrections for theories without boundary. We look at the effective action for the vector multiplet scalar after integrating out the chiral multiplet fields, this time with Wilson line boundary interaction. When the theory is formulated on a strip of width L, the result is [HHP08]:

$$U_{\text{eff}}(\sigma) = L\frac{e_{\text{eff}}^2}{2}|r_{\text{eff}} - i\theta_{\text{eff}}|^2 + 2\left[-\left(\frac{\theta}{2\pi} + q\right)\Re\sigma + \frac{s}{2}|\Re\sigma|\right], \tag{1.73}$$

where $r_{\text{eff}} = r + \sum_i Q_i \log|Q_i|$ as before, $s = \sum_{Q_i \geq 0} Q_i$ as before, and $\theta_{\text{eff}} = \theta - \operatorname{sgn}(\Re\sigma)s\pi$. The linear dependence on σ means that whenever there is a Coulomb branch, i.e. σ is unconstrained, the potential can become unbounded from below. Such potentials signal instability in a physical system since it appears that there is no stable vacuum. The potential becomes bounded from below and therefore the problem of instability is cured if the gauge charge of the Wilson line brane is constrained:

$$-\frac{s}{2} < \frac{\theta}{2\pi} + q < \frac{s}{2}. \tag{1.74}$$

Wilson line branes with charges satisfying this constraint can be safely transported through phase boundaries without becoming unstable. This criterion for stable brane transport in the Kähler moduli space was one of the key results of [HHP08], where the constraint (1.74) was termed the *grade restriction rule*.

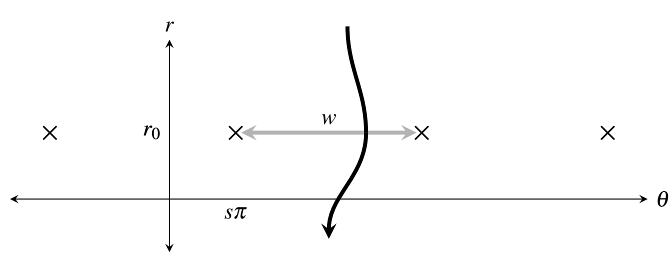

Fig. 1.3 Brane transport through a window in a Kähler moduli space. $r_0 = -\sum_i Q_i \log Q_i$, as in (1.23).

Since the theta angle θ is a periodic variable, we must specify a *window* in the complexified Kähler moduli space, through which we wish to transport a brane. Figure 1.3 shows such a window w and a schematic path to transport branes along. Given a window w, we define the set of gauge charges that satisfy the grade restriction rule (1.74):

$$N^w := \left\{ n \in \mathbb{Z} \,|\, \forall \theta \in w : -\frac{s}{2} < \frac{\theta}{2\pi} + q < \frac{s}{2} \right\} . \tag{1.75}$$

A GLSM B-brane with charge q will be called *grade restricted* with respect to a window w if $q \in N^w$. The set of all grade restricted brane with respect to a window w will be called $\mathscr{T}^w$.

The main goal is to find a one to one correspondence between IR branes in different phases. We find this correspondence by transporting branes across phase boundaries. The general prescription for doing so is the following: Suppose we are given a window w between two adjacent phases, one containing $r \in \mathscr{M}_K$ and the other $r' \in \mathscr{M}_K$. Then given an IR brane $\mathscr{B}_{\mathrm{IR}} \in \mathrm{D}(X_r)$ in the phase containing r, we first lift it to a GLSM brane $\mathscr{B} \in \mathfrak{D}(\mathbb{C}^N, T)$ satisfying $\pi_r(\mathscr{B}) = \mathscr{B}_{\mathrm{IR}}$. If $\mathscr{B} \notin \mathscr{T}^w$, then we find another brane $\mathscr{B}'$ such that $\mathscr{B}' \in \mathscr{T}^w$ and $\pi_r(\mathscr{B}')$ is quasi-isomorphic to $\pi_r(\mathscr{B})$. Now we can transport this brane $\mathscr{B}'$ across the window and then project down in the adjacent phase with the outcome $\pi_{r'}(\mathscr{B}')$.

Given any two adjacent phases containing $r, r' \in \mathscr{M}_K$ and a window w between them, there is a unique lift from $\mathrm{D}(X_r)$ and $\mathrm{D}(X_{r'})$ to $\mathscr{T}^w$ [HHP08] which we will denote as $\omega^w_{r,r'}$ and $\omega^w_{r',r}$ respectively. We can draw all the relevant maps in the following diagram:

$$\begin{tikzcd} & \mathfrak{D}(\mathbb{C}^N, T) \arrow[dl, "\pi_r"'] \arrow[dr, "\pi_{r'}"] & \\ \mathrm{D}(X_r) \arrow[r, "\omega^w_{r,r'}"'] & \mathscr{T}^w \arrow[u, hook] & \mathrm{D}(X_{r'}) \arrow[l, "\omega^w_{r',r}"] \end{tikzcd} \tag{1.76}$$

Brane transport from $\mathrm{D}(X_r)$ to $\mathrm{D}(X_{r'})$ is the composition:

$$\mathrm{D}(X_r) \xrightarrow{\omega^w_{r,r'}} \mathscr{T}^w \xrightarrow{\pi_{r'}} \mathrm{D}(X_{r'}) . \tag{1.77}$$

Transport in the other direction is defined similarly.

Example 3. We illustrate this mechanism of brane transport in the context of our example $U(1)$ gauge theory (see example 1, 2). The singularities in the Kähler moduli space are at $r = 3\log 3$ and $\theta \in 3\pi + 2\pi\mathbb{Z}$. In the geometric phase $r > 0$, where branes are represented by holomorphic bundles (coherent sheaves to be more precise) we pick the brane[41] $\underline{\mathscr{O}(2)}$ and we wish to transport it from the phase $r > 0$ to the orbifold phase $r < 0$ through the window $w : -\pi < \theta < \pi$. The obvious

[41] We use an underline to point out the R-charge (differential degree) zero part of a complex.

lift of this brane to a GLSM brane is $\mathscr{W}(2)$. But note that in the window w the grade restricted gauge charges are $N^w = \{-1,0,1\}$ so the GLSM brane $\mathscr{W}(2)$ is not grade restricted. On the other hand in example 2 we saw that the brane $\mathscr{C}(\mathscr{B}_1) : \mathscr{W}^{-2}(-1) \xrightarrow{d_{-3}} \mathscr{W}^{-1}(0)^{\oplus 3} \xrightarrow{d_{-2}} \mathscr{W}^{0}(1)^{\oplus 3}$ is quasi-isomorphic to $\mathscr{W}^0(2)$ at low energy in the phase $r > 0$, and $\mathscr{B}_1$ is grade restricted. Therefore, we can transport $\mathscr{B}_1$ over to the phase $r < 0$ and project down to $\pi_-(\mathscr{B}_1)$:

$$\pi_-(\mathscr{B}_1) : \mathscr{O}(-1) \xrightarrow{d_{-3}} \mathscr{O}(0)^{\oplus 3} \xrightarrow{d_{-2}} \underline{\mathscr{O}(1)}^{\oplus 3} . \tag{1.78}$$

The above brane is the image in $\mathrm{D}(X_-)$ of $\underline{\mathscr{O}(2)} \in \mathrm{D}(X_+)$ under brane transport.

More generally, consider a GLSM with abelian gauge group T that reduces, in a phase, to an orbifold theory $X_{\text{orb}} = \mathbb{C}^N/\Gamma$ with a discrete gauge group $\Gamma \subset T$. The GLSM Chan-Paton space reduces to a representation of the discrete gauge group Γ and the low energy branes in the orbifold phase are complexes of Γ-equivariant vector bundles on $\mathbb{C}^N$, $\mathrm{D}(X_{\text{orb}}) \cong \mathrm{D}_\Gamma(\mathbb{C}^N)$. Other phases of this GLSM are given by partial or complete *crepant resolutions* of the orbifold singularity. If we denote such a resolution as X_{res}, then the brane transport establishes a correspondence of D-branes between these phases:

$$\mathrm{D}_\Gamma(\mathbb{C}^N) \quad \xleftrightarrow{\text{brane transport}} \quad \mathrm{D}(X_{\text{res}}) . \tag{1.79}$$

Considering the mapping between the chiral sectors, this becomes an equivalence of derived categories which is known as *McKay correspondence* [Kaw03, BKR99].

In more general cases, with non-zero superpotential, complexes of sheaves are replaced by matrix factorizations of the superpotential and low energy computation becomes more involved as some of the fields can acquire masses from the superpotential and need to be integrated out. The main ideas remain unchanged. The low energy branes are still the GLSM branes upto phase dependent quasi-isomorphisms. The grade restriction rules are the same and brane transport works similarly. In the math literature, the relevant equivalence of derived categories, that of CY hypersurface defined by a polynomial and of matrix factorization of the same polynomial, was established by Orlov [Orl05]. Details including relation between the physical perspective of brane transport and Orlov's construction of categorical equivalence can be found in [HHP08].

1.A $\mathscr{N} = (2,2)$ Supersymmetry

Our world-sheet has, as time and space coordinates, y^0 and y^1 respectively, with a Lorentzian signature $\mathrm{d}s^2 = -(\mathrm{d}y^0)^2 + (\mathrm{d}y^1)^2$. It is convenient to introduce *light cone* coordinates and their derivatives:

$$z^\pm := y^0 \pm y^1 , \qquad \partial_\pm := \frac{\partial}{\partial z^\pm} = \frac{1}{2}\left(\partial_0 \pm \partial_1\right) . \tag{1.80}$$

The $\mathcal{N} = (2,2)$ supersymmetry algebra contains four supercharges:

$$\boldsymbol{Q}_+ \,, \boldsymbol{Q}_- \,, \overline{\boldsymbol{Q}}_+ \,, \overline{\boldsymbol{Q}}_- \,, \tag{1.81}$$

and the following bosonic generators:

$$\text{Time translation} : H := \partial_0 \,, \tag{1.82a}$$
$$\text{Spatial translation} : P := \partial_1 \,, \tag{1.82b}$$
$$\text{Rotation} : M := x^0 \partial_1 + x^1 \partial_0 \,. \tag{1.82c}$$

Depending on the field theory, this algebra may be augmented by two $U(1)$ R-symmetry generators:

$$F_V \,, F_A, , \tag{1.83}$$

corresponding to the vector R-symmetry ($U(1)_V$) and the axial R-symmetry ($U(1)_A$) respectively. The non-zero commutation relations are:

$$\{\boldsymbol{Q}_\pm, \overline{\boldsymbol{Q}}_\pm\} = H \pm P \,, \tag{1.84a}$$
$$[iM, \boldsymbol{Q}_\pm] = \mp \boldsymbol{Q}_\pm \,, \quad [iM, \overline{\boldsymbol{Q}}_\pm] = \mp \overline{\boldsymbol{Q}}_\pm \,, \tag{1.84b}$$
$$[F_V, \boldsymbol{Q}_\pm] = -\boldsymbol{Q}_\pm \,, \quad [F_V, \overline{\boldsymbol{Q}}_\pm] = \overline{\boldsymbol{Q}}_\pm \,, \tag{1.84c}$$
$$[F_A, \boldsymbol{Q}_\pm] = \mp \boldsymbol{Q}_\pm \,, \quad [F_A, \overline{\boldsymbol{Q}}_\pm] = \pm \overline{\boldsymbol{Q}}_\pm \,. \tag{1.84d}$$

The hermiticity property of the supercharges is:

$$\boldsymbol{Q}_\pm^\dagger = \overline{\boldsymbol{Q}}_\pm \,. \tag{1.85}$$

The bosonic operators are all hermitian.

Two distinguished subalgebras of the above algebra containing half of the supersymmetry are defined with the following supercharges:

$$\mathcal{N} = 2_A : \qquad \boldsymbol{Q}_A \,, \boldsymbol{Q}_A^\dagger \,, \qquad \boldsymbol{Q}_A := \overline{\boldsymbol{Q}}_+ + \boldsymbol{Q}_- \,, \tag{1.86a}$$
$$\mathcal{N} = 2_B : \qquad \boldsymbol{Q}_B \,, \boldsymbol{Q}_B^\dagger \,, \qquad \boldsymbol{Q}_B := \overline{\boldsymbol{Q}}_+ + \overline{\boldsymbol{Q}}_- \,. \tag{1.86b}$$

They satisfy the same anti-commutation relations, for $\boldsymbol{Q} \in \{\boldsymbol{Q}_A, \boldsymbol{Q}_B\}$:

$$\{\boldsymbol{Q}, \boldsymbol{Q}^\dagger\} = 2H \,, \qquad \boldsymbol{Q}^2 = \boldsymbol{Q}^{\dagger^2} = 0 \,. \tag{1.87}$$

Their charges under the R-symmetries play important role in our analysis:

$$[F_V, \boldsymbol{Q}_A] = 0 \,, \qquad [F_A, \boldsymbol{Q}_A] = Q_A \,, \tag{1.88a}$$
$$[F_V, \boldsymbol{Q}_B] = Q_B \,, \qquad [F_A, \boldsymbol{Q}_B] = 0 \,, \tag{1.88b}$$

which implies that a $\boldsymbol{Q}$-complex (computing some cohomology of interest) will be $\mathbb{Z}$-graded by the $U(1)_V$ for $\boldsymbol{Q} = \boldsymbol{Q}_B$ and $U(1)_A$ for $\boldsymbol{Q} = \boldsymbol{Q}_A$, given that these R-symmetries are preserved by the quantum theory.

1.B Chan-Paton space

Let us consider a general setup of having a QFT on a world-sheet (a surface) Σ with boundary. We may have a set of possible boundary conditions[42] that we can assign to the fields of the theory, let us call this set of *basic boundary conditions* B_0. Each connected component of the boundary $\partial\Sigma$ corresponds to a brane (to which the component is thought to be attached) or equivalently, each boundary condition corresponds to a brane. If we have a configuration of multiple branes with a boundary condition for each of them then we can label each brane with an index, let us call the index set $\mathscr{I}$, and collect all these boundary conditions in the map $\mathscr{B} : \mathscr{I} \to B_0$. The indices in the set $\mathscr{I}$ are called Chan-Paton indices. This way each end point of the strings will carry an index of its own. More generally it may be possible, and times necessary, to consider linear combinations of indices attached to an end point, or to allow some group to act on them.

For example, if we have N coincident identical branes then there may be a symmetry rotating those branes (among each other). Once we introduce an index for each brane, $\mathscr{I} = \{1, \cdots, N\}$, we can define the map $\mathscr{B} : \mathscr{I} \to B_0$ to send each index to the same boundary condition. Now some rotation subgroup[43] of $GL(\mathbb{C}^N)$ acting on the indices in $\mathscr{I}$ can become a symmetry of the theory. Another way to interpret this is to imagine, instead of a stack of N branes, just a single brane with a vector bundle like structure where the fiber over each point is $\mathbb{C}^N$. Then an open string that ends on this brane has not only a specific position on the brane attached to its end point but also a specific position along the fiber and some subgroup of $GL(\mathbb{C}^N)$ can act on this position along the fiber. In this picture it is said that the end points of the open strings always have a vector space attached to them, the vector spaces being the fibers over the points on the branes where the end points are attached. These vector spaces are called Chan-Paton spaces, we will use the letter $\mathscr{V}$ to refer to Chan-Paton spaces.

Let us schematically discuss some fairly general properties of open string Hilbert spaces in the presence of non-trivial Chan-Paton spaces. We will parametrize time and space with y^0 and y^1 respectively, then in the presence of two boundaries at $y^1 = 0$ and $y^1 = L$, computing correlation functions on the semi-infinite strip $[0, L] \times [-\infty, t]$ defines wave functions in the Hilbert space associated to the open string at time t (see figure 1.4). To do the path integral we have to use two boundary conditions for the two boundaries and specify a value for the fields at time t. These boundary conditions can include different boundary actions to preserve some symmetry. Corresponding to two boundary conditions $\mathscr{B}(i)$ and $\mathscr{B}(j)$ for $i, j \in \mathscr{I}$, let us choose two boundary actions $I_i(y^1 = 0) := \int_{-\infty}^{t} \mathrm{d}y^0\, \mathscr{L}_i$ and $I_j(y^1 = L) := \int_{-\infty}^{t} \mathrm{d}y^0\, \mathscr{L}_j$ supported on $y^1 = 0$ and $y^1 = L$ respectively. Now a state in the Hilbert space at time

[42] A boundary condition can involve literally boundary conditions for the fields at the boundary along with boundary actions supported on the boundary that help preserve some of the symmetries of the bulk theory.

[43] We are using the term "rotation" loosely, the actual symmetry depends on the details of the theory, it can be $U(N)$, $SO(N)$, $Sp(N)$, etc.

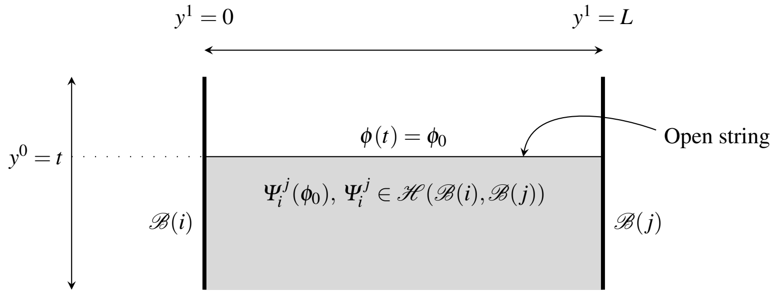

Fig. 1.4 Wave function on an open string with two boundaries.

t is defined by the path integral with some operator insertion:

$$\Psi_i^j(\phi_0) := \int_{\phi(t)=\phi_0} \mathscr{D}\phi\, \mathscr{O}_\Psi(\phi) e^{i(S+I_i(y^1=0)+I_j(y^1=L))}\,, \tag{1.89}$$

where the path integral is over all field configurations that take the specific value ϕ_0 at time t and satisfy some boundary conditions (such as Dirichlet, Neuman etc.) at the boundaries. We have put the indices at different heights to keep track of the orientation of the string. The Hilbert space consisting of all such wave functions for fixed i and j will be called $\mathscr{H}(\mathscr{B}(i), \mathscr{B}(j))$. The total open string Hilbert space is a direct sum over all possible boundary conditions:

$$\mathscr{H} = \bigoplus_{i,j \in \mathscr{I}} \mathscr{H}(\mathscr{B}(i), \mathscr{B}(j))\,. \tag{1.90}$$

Since the indices in $\mathscr{I}$ can be thought to represent the basis vectors of the Chan-Paton space $\mathscr{V}$, in an index free notation the wave function Ψ_i^j becomes valued in $\mathscr{V}^* \otimes \mathscr{V} = \mathrm{Hom}(\mathscr{V}, \mathscr{V})$.[44] More generally, we can consider two different configurations of branes for the two end points of an open string. Equivalently, we can have two different index sets and sets of basic boundary conditions, i.e., two different Chan-Paton spaces, $\mathscr{V}_1$ and $\mathscr{V}_2$, for the two ends. In such cases the open string states will be valued in $\mathrm{Hom}(\mathscr{V}_1, \mathscr{V}_2)$.

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[44] The dual distinguishes the lower index from the upper in terms of representations of groups acting on $\mathscr{V}$. This is important for oriented strings, for unoriented strings the relevant representations must be self-dual.

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